

# Rank-3 Projections of a 4-Cube

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**ABSTRACT.** The orthogonal projection of a 4-cube onto a uniform random 3-subspace in  $\mathbb{R}^4$  is a convex 3-polyhedron  $P$  with 14 vertices almost surely. Three numerical characteristics of  $P$  – volume, surface area and mean width – are studied. These quantities, along with the Euler characteristic, form a basis of the space of all additive continuous measures that are invariant under rigid motions in  $\mathbb{R}^3$ . While computing statistics of  $\{vl, ar, mw\}$ , we encounter the generalized hypergeometric function, elliptic integrals and Catalan's constant. A new constant 7.1185587167... also arises and deserves further attention.

A planar shadow of a 3-cube {4-cube} is a convex hexagon {octagon} almost surely. In an earlier paper [1], joint moments of hexagonal {octagonal} area and perimeter were computed. It is natural to speculate on  $n$ -cubes, under the action of corank-1 projections rather than rank-2.

Let  $C$  denote a 4-cube with edges of unit length, centered at the origin. To generate a random 3-subspace  $S$  in  $\mathbb{R}^4$ , we select a random point  $U$  uniformly on the 3-sphere of unit radius. The desired subspace is the set of all vectors orthogonal to  $U$ .

We then project the (fixed) 4-cube  $C$  orthogonally onto  $S$ . This is done by forming the convex hull of images of all vertices of  $C$ . The resultant polyhedron in the hyperplane has 14 vertices almost surely. More precisely, if  $U = (x, y, z, w)$  is of unit length, then the matrix

$$M_4 = \begin{pmatrix} \sqrt{1-x^2} & -\frac{xy}{\sqrt{1-x^2}} & -\frac{xz}{\sqrt{1-x^2}} & -\frac{xw}{\sqrt{1-x^2}} \\ 0 & \sqrt{\frac{z^2+w^2}{1-x^2}} & -\frac{yz}{\sqrt{(1-x^2)(z^2+w^2)}} & -\frac{yw}{\sqrt{(1-x^2)(z^2+w^2)}} \\ 0 & 0 & \frac{w}{\sqrt{z^2+w^2}} & -\frac{z}{\sqrt{z^2+w^2}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

projects  $C$  orthogonally onto a hyperplane, rotated in  $\mathbb{R}^4$  to coincide with the 3-subspace spanned by  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$  and  $(0, 0, 1, 0)$  for convenience. Let  $T$  be the 4-row matrix whose columns constitute all vertices of  $C$ . Then the first 3 rows of  $M_4 T$  constitute all images of the vertices in  $\mathbb{R}^3$  and 3-dimensional convex hull algorithms apply naturally. Such calculations provide the underpinning for our work.

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Spherical coordinates in  $\mathbb{R}^4$ :

$$x = \cos \theta \sin \varphi \sin \psi, \quad y = \sin \theta \sin \varphi \sin \psi, \quad z = \cos \varphi \sin \psi, \quad w = \cos \psi$$

will be used throughout for  $U$ , where  $0 \leq \theta < 2\pi$ ,  $0 \leq \varphi \leq \pi$ ,  $0 \leq \psi \leq \pi$ . The corresponding Jacobian determinant is  $\sin \varphi \sin^2 \psi$ ; it is best to think of  $(\theta, \varphi, \psi)$  as possessing joint density  $\frac{1}{2\pi^2} \sin \varphi \sin^2 \psi$ .

### 1. VOLUME

The projected polyhedral volume  $vl$  in  $\mathbb{R}^3$  is equal to

$$|x| + |y| + |z| + |w|,$$

given a unit vector  $U = (x, y, z, w)$ . It follows that

$$\begin{aligned} \mathbb{E}(vl) &= 4 \cdot 16 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos \psi \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi \, d\psi \, d\varphi \, d\theta \\ &= \frac{16}{3\pi} = 1.697652726313550... \end{aligned}$$

$$\begin{aligned} \mathbb{E}(vl^2) &= 4 \cdot 16 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 \psi \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi \, d\psi \, d\varphi \, d\theta \\ &\quad + 12 \cdot 16 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos \varphi \sin \psi \cos \psi \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi \, d\psi \, d\varphi \, d\theta \\ &= 1 + \frac{6}{\pi} = 2.909859317102744... \end{aligned}$$

More generally, starting with a unit  $n$ -cube, the projected  $(n-1)$ -polyhedral volume  $vl$  in  $\mathbb{R}^{n-1}$  satisfies

$$\begin{aligned} \mathbb{E}(vl) &= \frac{n}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}, \\ \mathbb{E}(vl^2) &= 1 + \frac{2(n-1)}{\pi}. \end{aligned}$$

These volume moment formulas are the same as formulas in [2] for the mean width and mean square width, respectively, of the  $n$ -cube itself. Such duality is a special case of a theorem proved in [3, 4, 5, 6]. We also have  $\max(vl) = \sqrt{n}$  and  $\min(vl) = 1$ .

## 2. SURFACE AREA

The projected polyhedral surface area  $ar$  in  $\mathbb{R}^3$  is equal to

$$2 \left( \sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} + \sqrt{x^2 + w^2} + \sqrt{y^2 + z^2} + \sqrt{y^2 + w^2} + \sqrt{z^2 + w^2} \right),$$

given a unit vector  $U = (x, y, z, w)$ . It follows that

$$\mathbb{E}(ar) = 6 \cdot 16 \cdot 2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\cos^2 \varphi \sin^2 \psi + \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta = 8$$

and, since  $x^2 + y^2 = \sin^2 \varphi \sin^2 \psi$ ,

$$\begin{aligned} \mathbb{E}(ar^2) &= 6 \cdot 16 \cdot 4 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} (\cos^2 \varphi \sin^2 \psi + \cos^2 \psi) \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta \\ &\quad + 24 \cdot 16 \cdot 4 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sin \varphi \sin \psi \sqrt{\sin^2 \theta \sin^2 \varphi \sin^2 \psi + \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta \\ &\quad + 6 \cdot 16 \cdot 4 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sin \varphi \sin \psi \sqrt{\cos^2 \varphi \sin^2 \psi + \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta \\ &= 12 + 6\zeta_4 + 3\pi = 64.136130261087789\dots \end{aligned}$$

Details on derivation of the mean square result, including a formula for the constant  $\zeta_4$ , will appear shortly. We also have  $\max(ar) = 6\sqrt{2}$  and  $\min(ar) = 6$ .

More generally, starting with a unit  $n$ -cube, the projected  $(n-1)$ -polyhedral surface area  $ar$  in  $\mathbb{R}^{n-1}$  satisfies

$$\mathbb{E}(ar) = \frac{\sqrt{\pi}(n-1)n}{2} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})},$$

$$\mathbb{E}(ar^2) = 4(n-1) + (n-2)(n-1)\zeta_n + \frac{(n-3)(n-2)(n-1)}{2}\pi.$$

For the special case  $n = 3$ , we have

$$\frac{ar}{2} = \sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} + \sqrt{y^2 + z^2} = \sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2} = \frac{\pi}{2} mw$$

given that  $x^2 + y^2 + z^2 = 1$ , which implies a closed-form expression for  $\zeta_3$  (using results from an upcoming subsection “2D Analog”). Numerical evidence strongly suggests that  $\zeta_n = \zeta_3$  for all  $n \geq 4$ , but a rigorous proof is not known.

## 3. MEAN WIDTH

The projected polyhedral mean width  $mw$  in  $\mathbb{R}^3$  is equal to

$$\frac{1}{2} \left( \sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2} + \sqrt{1-w^2} \right),$$

given a unit vector  $U = (x, y, z, w)$ . It follows that

$$\begin{aligned} \mathbb{E}(mw) &= 4 \cdot 16 \cdot \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{1 - \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi \, d\psi \, d\varphi \, d\theta \\ &= \frac{16}{3\pi} = \mathbb{E}(vl), \end{aligned}$$

$$\begin{aligned} \mathbb{E}(mw^2) &= 4 \cdot 16 \cdot \frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} (1 - \cos^2 \psi) \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi \, d\psi \, d\varphi \, d\theta \\ &\quad + 12 \cdot 16 \cdot \frac{1}{4} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{1 - \cos^2 \varphi \sin^2 \psi} \sqrt{1 - \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi \, d\psi \, d\varphi \, d\theta \\ &= 3 \left( \frac{1}{4} + \frac{\pi}{8} + \frac{1}{\pi} \right) = 2.883026903647544... < 1 + \frac{6}{\pi} = \mathbb{E}(vl^2). \end{aligned}$$

We also have  $\max(mw) = \sqrt{3}$  and  $\min(mw) = 3/2$ .

More generally, starting with a unit  $n$ -cube, the projected  $(n-1)$ -polyhedral mean width  $mw$  in  $\mathbb{R}^{n-1}$  satisfies

$$\mathbb{E}(mw) = \frac{n}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \mathbb{E}(vl)$$

but a general formula for  $\mathbb{E}(mw^2)$  is unknown. We examine the cases  $n = 3$  and  $n = 5$  in the following subsections.

**3.1. 2D Analog.** As outlined in [1],

$$mw = \frac{2}{\pi} \left( \sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2} \right)$$

given that  $x^2 + y^2 + z^2 = 1$ , hence  $\mathbb{E}(mw) = 3/2$  and

$$\begin{aligned} \mathbb{E}(mw^2) &= 3 \cdot 8 \cdot \left( \frac{2}{\pi} \right)^2 \int_0^{\pi/2} \int_0^{\pi/2} (1 - \cos^2 \varphi) \frac{1}{4\pi} \sin \varphi \, d\varphi \, d\theta \\ &\quad + 6 \cdot 8 \cdot \left( \frac{2}{\pi} \right)^2 \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{1 - \cos^2 \theta \sin^2 \varphi} \sqrt{1 - \cos^2 \varphi} \frac{1}{4\pi} \sin \varphi \, d\varphi \, d\theta \\ &= \frac{2}{\pi^2} \left[ 4 + 3\pi {}_3F_2 \left( -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; 1 \right) \right] = 2.253091059149751... < 1 + \frac{4}{\pi} \end{aligned}$$

where

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\Gamma(a_2+k)\Gamma(a_3+k)}{\Gamma(b_1+k)\Gamma(b_2+k)} \frac{z^k}{k!}$$

is the generalized hypergeometric function. As a byproduct, the constant  $\zeta_3$  defined earlier is equal to  $3\pi {}_3F_2 \left( -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; 1 \right)$ . We also have  $\max(mw) = 2\sqrt{6}/\pi$  and  $\min(mw) = 4/\pi$ .

**3.2. 4D Analog.** We here have

$$mw = \frac{4}{3\pi} \sum_{j=1}^5 \sqrt{1 - x_j^2}$$

given that  $\sum_{j=1}^5 x_j^2 = 1$ , hence  $\mathbb{E}(mw) = 15/8$  and

$$\mathbb{E}(mw^2) = 32 \cdot \left(\frac{4}{3\pi}\right)^2 (5I + 20J)$$

where

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} (1 - \cos^2 \varphi_3) \frac{3}{8\pi^2} \sin \varphi_1 \sin^2 \varphi_2 \sin^3 \varphi_3 d\varphi_3 d\varphi_2 d\varphi_1 d\theta,$$

$$J = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{1 - \cos^2 \varphi_2 \sin^2 \varphi_3} \sqrt{1 - \cos^2 \varphi_3} \frac{3}{8\pi^2} \sin \varphi_1 \sin^2 \varphi_2 \sin^3 \varphi_3 d\varphi_3 d\varphi_2 d\varphi_1 d\theta.$$

The expression for  $\mathbb{E}(mw^2)$  simplifies to

$$\begin{aligned} & \frac{4}{81\pi^4} \left[ 144\pi^2 - 10\Gamma\left(\frac{1}{4}\right)^4 + 45\pi^3 \left( {}_3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; 1\right) - {}_3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 3; 1\right) \right) \right] \\ &= 3.516040901689803... < 1 + \frac{8}{\pi}. \end{aligned}$$

We also have  $\max(mw) = 8\sqrt{5}/(3\pi)$  and  $\min(mw) = 16/(3\pi)$ . The fact that  $\mathbb{E}(mw)$  at one level becomes  $\min(mw)$  at the next level is interesting.

## 4. DETAILS

With regard to  $\mathbb{E}(ar^2)$ , we first prove that

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sin \varphi \sin \psi \sqrt{\cos^2 \varphi \sin^2 \psi + \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta = \frac{\pi}{128}.$$

After rearranging and integrating out  $\theta$ , the integral becomes

$$\begin{aligned} & \frac{\pi}{2} \int_0^{\pi/2} \cos \psi \sin^3 \psi \int_0^{\pi/2} \sqrt{\cos^2 \varphi \tan^2 \psi + 1} \frac{1}{2\pi^2} \sin^2 \varphi d\varphi d\psi \\ &= \frac{1}{4\pi} \int_0^{\pi/2} \cos \psi \sin^3 \psi \frac{(-1 + \tan^2 \psi) E(i \tan \psi) + (1 + \tan^2 \psi) K(i \tan \psi)}{3 \tan^2 \psi} d\psi \end{aligned}$$

where  $i$  is the imaginary unit and

$$K(\xi) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \xi^2 \sin^2(\theta)}} d\theta, \quad E(\xi) = \int_0^{\pi/2} \sqrt{1 - \xi^2 \sin^2(\theta)} d\theta$$

are complete elliptic integrals of the first and second kind. Let  $t = \tan \psi$ , then

$$dt = \sec^2 \psi d\psi = (1 + t^2) d\psi$$

hence

$$\begin{aligned} \frac{\cos \psi \sin^3 \psi}{\tan^2 \psi} d\psi &= \cos^3 \psi \sin \psi d\psi = \left( \frac{1}{1 + t^2} \right)^{3/2} \left( 1 - \frac{1}{1 + t^2} \right)^{1/2} d\psi \\ &= \left( \frac{1}{1 + t^2} \right)^{5/2} \left( 1 - \frac{1}{1 + t^2} \right)^{1/2} dt \\ &= \frac{t}{(1 + t^2)^3} dt \end{aligned}$$

and the integral becomes

$$\begin{aligned} & \frac{1}{12\pi} \int_0^\infty \frac{(-1 + t^2) E(it) + (1 + t^2) K(it)}{(1 + t^2)^3} t dt \\ &= \frac{1}{12\pi} \left[ \int_0^\infty \frac{E(it)}{(1 + t^2)^2} t dt - 2 \int_0^\infty \frac{E(it)}{(1 + t^2)^3} t dt + \int_0^\infty \frac{K(it)}{(1 + t^2)^2} t dt \right] \\ &= \frac{\pi}{96} - \frac{\pi}{128} + \frac{\pi}{192} = \frac{\pi}{128}. \end{aligned}$$

The other integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sin \varphi \sin \psi \sqrt{\sin^2 \theta \sin^2 \varphi \sin^2 \psi + \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta$$

is harder. We rearrange it as

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\pi/2} \cos \psi \sin^3 \psi \int_0^{\pi/2} \sqrt{\sin^2 \theta \sin^2 \varphi \tan^2 \psi + 1} \frac{1}{2\pi^2} \sin^2 \varphi d\theta d\varphi d\psi \\ &= \frac{1}{2\pi^2} \int_0^{\pi/2} \cos \psi \sin^3 \psi \int_0^{\pi/2} E(i \sin \varphi \tan \psi) \sin^2 \varphi d\varphi d\psi \\ &= \frac{1}{2\pi^2} \int_0^{\pi/2} \cos \psi \sin^3 \psi \frac{f(\psi) + g(\psi) + h(\psi)}{3 \tan^2 \psi} d\psi, \end{aligned}$$

where

$$\begin{aligned} f(\psi) &= (-2 + 4 \tan^2 \psi) E \left( i \sqrt{\frac{\sec \psi - 1}{2}} \right)^2, \\ g(\psi) &= 2 (1 - 2 \tan^2 \psi + \sec \psi) E \left( i \sqrt{\frac{\sec \psi - 1}{2}} \right) K \left( i \sqrt{\frac{\sec \psi - 1}{2}} \right), \\ h(\psi) &= (-1 + \tan^2 \psi - \sec \psi) K \left( i \sqrt{\frac{\sec \psi - 1}{2}} \right)^2. \end{aligned}$$

Let  $t = \sqrt{(\sec \psi - 1)/2}$ , then  $1 + 2t^2 = \sec \psi$  and

$$4t dt = \sec \psi \tan \psi d\psi = (1 + 2t^2) \left( (1 + 2t^2)^2 - 1 \right)^{1/2} d\psi$$

hence

$$\begin{aligned} \frac{\cos \psi \sin^3 \psi}{\tan^2 \psi} d\psi &= \cos^3 \psi \sin \psi d\psi = \left( \frac{1}{1 + 2t^2} \right)^3 \left( 1 - \frac{1}{(1 + 2t^2)^2} \right)^{1/2} d\psi \\ &= (4t) \left( \frac{1}{1 + 2t^2} \right)^4 \left( 1 - \frac{1}{(1 + 2t^2)^2} \right)^{1/2} \frac{1}{((1 + 2t^2)^2 - 1)^{1/2}} dt \\ &= \frac{4t}{(1 + 2t^2)^5} dt. \end{aligned}$$



The integral becomes

$$\begin{aligned} & \frac{2}{3\pi^2} \int_0^\infty \frac{\left[-2 + 4 \left((1 + 2t^2)^2 - 1\right)\right] E(it)^2}{(1 + 2t^2)^5} t dt \\ & + \frac{4}{3\pi^2} \int_0^\infty \frac{\left[1 - 2 \left((1 + 2t^2)^2 - 1\right) + (1 + 2t^2)\right] E(it) K(it)}{(1 + 2t^2)^5} t dt \\ & + \frac{2}{3\pi^2} \int_0^\infty \frac{\left[-1 + \left((1 + 2t^2)^2 - 1\right) - (1 + 2t^2)\right] K(it)^2}{(1 + 2t^2)^5} t dt \end{aligned}$$

which simplifies to

$$\frac{4}{3\pi^2} \int_0^\infty \frac{(8t^4 + 8t^2 - 1) E(it)^2 - 2(4t^4 + 3t^2 - 1) E(it) K(it) + (2t^4 + t^2 - 1) K(it)^2}{(2t^2 + 1)^5} t dt.$$

Call this expression  $\zeta_4/256$ . The constant  $\zeta_4 = 7.118558716719735\dots$  coincides with  $\zeta_3$  to high numerical precision, but algebraic confirmation remains open.

## 5. CORRELATIONS

The joint moment for volume and surface area is

$$\begin{aligned} \mathbb{E}(vl \cdot ar) &= 12 \cdot 16 \cdot 2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \left[ \cos \psi \sqrt{\cos^2 \varphi \sin^2 \psi + \cos^2 \psi} \right. \\ & \quad \left. + \cos \psi \sqrt{\sin^2 \theta \sin^2 \varphi \sin^2 \psi + \cos^2 \varphi \sin^2 \psi} \right] \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta \\ &= 6 \left( 1 + \frac{4}{\pi} \right) = 13.639437268410976\dots \end{aligned}$$

which, together with earlier results, implies that the correlation between  $vl$  and  $ar$  is 0.945.... Likewise, we have

$$\begin{aligned} \mathbb{E}(vl \cdot mw) &= 4 \cdot 16 \cdot \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos \psi \sqrt{1 - \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta \\ & \quad + 12 \cdot 16 \cdot \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos \varphi \sin \psi \sqrt{1 - \cos^2 \psi} \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta \\ &= \frac{9}{4} + \frac{2}{\pi} = 2.886619772367581\dots, \end{aligned}$$

$$\begin{aligned}
\mathbb{E}(ar \cdot mw) &= 12 \cdot 16 \cdot 2 \cdot \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \left[ \sqrt{\cos^2 \varphi \sin^2 \psi + \cos^2 \psi} \sqrt{1 - \cos^2 \psi} \right. \\
&\quad \left. + \sqrt{\sin^2 \theta \sin^2 \varphi \sin^2 \psi + \cos^2 \varphi \sin^2 \psi} \sqrt{1 - \cos^2 \psi} \right] \frac{1}{2\pi^2} \sin \varphi \sin^2 \psi d\psi d\varphi d\theta \\
&= \frac{3(5 + 2G)}{\pi} + \frac{9\pi}{4} = 13.592597187518807...
\end{aligned}$$

where

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

is Catalan's constant [7]. The correlation between  $vl$  and  $mw$  is consequently 0.870... and the correlation between  $ar$  and  $mw$  is 0.973....

## 6. RELATED WORK

Let us return to an open issue in [1] surrounding rank-2 projections of a 4-cube. On the one hand, octagonal perimeter is equal to

$$2 \left( \sqrt{1 - p^2 - x^2} + \sqrt{1 - q^2 - y^2} + \sqrt{1 - r^2 - z^2} + \sqrt{1 - s^2 - w^2} \right),$$

given orthogonal unit vectors  $U = (x, y, z, w)$  and  $V = (p, q, r, s)$ . This formula makes accurate calculation of the second moment of perimeter 28.495... more feasible. The maximum value of perimeter is  $4\sqrt{2}$ ; the minimum value is 4.

On the other hand, octagonal area does not appear to have so simple a description. Let

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \cos \kappa \sin \lambda \begin{pmatrix} -y \\ x \\ -w \\ z \end{pmatrix} + \sin \kappa \sin \lambda \begin{pmatrix} -z \\ w \\ x \\ -y \end{pmatrix} + \cos \lambda \begin{pmatrix} -w \\ -z \\ y \\ x \end{pmatrix}$$

where  $0 \leq \kappa < 2\pi$ ,  $0 \leq \lambda \leq \pi$ . Let

$$a_1 = ry + sy - qz - sz - qw + rw,$$

$$a_2 = ry - sy - qz - sz + qw + rw,$$

$$a_3 = ry + sy - qz + sz - qw - rw,$$

$$b_1 = pq - ps + xy - xw,$$

$$b_2 = pq - pr + xy - xz,$$

$$b_3 = p q + p s + x y + x w,$$

$$c = 2(1 - p^2 - x^2).$$

Then, in a neighborhood of  $(\theta_0, \varphi_0, \psi_0, \kappa_0, \lambda_0) = (1, 1, 1, 1, 1)$ , octagonal area is equal to

$$\frac{1}{c} [a_1 b_2 + a_2(c - b_1 + b_2) + a_3 b_1];$$

in a neighborhood of  $(\theta_0, \varphi_0, \psi_0, \kappa_0, \lambda_0) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ , octagonal area is equal to

$$\frac{1}{c} [a_1 b_2 - a_2(b_1 - b_2) + a_3(c + b_1)];$$

in a neighborhood of  $(\theta_0, \varphi_0, \psi_0, \kappa_0, \lambda_0) = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ , octagonal area is equal to

$$\frac{1}{c} [a_1 b_2 - a_2(c + b_1 - b_2) + a_3 b_1];$$

in a neighborhood of  $(\theta_0, \varphi_0, \psi_0, \kappa_0, \lambda_0) = (\frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6})$ , octagonal area is equal to

$$\frac{1}{c} [-a_1(c - b_2) - a_2(b_1 - b_2) + a_3 b_1];$$

in a neighborhood of  $(\theta_0, \varphi_0, \psi_0, \kappa_0, \lambda_0) = (\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8})$ , octagonal area is equal to

$$\frac{1}{c} [a_1 b_2 - a_2(b_1 - b_2) - a_3(c - b_1)];$$

in a neighborhood of  $(\theta_0, \varphi_0, \psi_0, \kappa_0, \lambda_0) = (\frac{4}{5}, 1, \frac{2}{5}, \frac{3}{5}, \frac{1}{5})$ , octagonal area is equal to

$$\frac{1}{c} [a_1(c + b_2) - a_2(b_1 - b_2) + a_3 b_1];$$

Other branches also exist, but for reasons of space, we stop here. The maximum value of octagonal area is  $1 + \sqrt{2}$  [5, 6]; the minimum value is 1.

An addendum to [8] now exists, with voluminous detail on the derivation of a relevant density function [9].

The present work is the fifth (and final) in a pentalogy that began with [10], continued with [2, 11] and then again with [1].

## 7. ACKNOWLEDGEMENTS

Wouter Meeussen's package ConvexHull3D.m was helpful to me in preparing this paper [12]. He kindly extended the software functionality at my request. Much more relevant material can be found at [13], including experimental computer runs that aided theoretical discussion here.

## 8. ADDENDUM

Here is a proof that  $\zeta_5 = \zeta_4$ . For  $n = 5$ , we have

$$ar = 2 \sum_{1 \leq j < k \leq 5} \sqrt{x_j^2 + x_k^2}$$

given that  $\sum_{j=1}^5 x_j^2 = 1$ . The hard integral for  $n = 4$  in a preceding subsection “Details” here becomes

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \sqrt{\sin^2 \theta \sin^2 \varphi_1 \sin^2 \varphi_2 \sin^2 \varphi_3 + \cos^2 \varphi_3} \\ & \quad \times \frac{3}{8\pi^2} \sin \varphi_1 \sin^2 \varphi_2 \sin^3 \varphi_3 d\varphi_3 d\varphi_2 d\varphi_1 d\theta \\ = & \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos \varphi_3 \sin^4 \varphi_3 \int_0^{\pi/2} \sqrt{\sin^2 \theta \sin^2 \varphi_1 \sin^2 \varphi_2 \tan^2 \varphi_3 + 1} \\ & \quad \times \frac{3}{8\pi^2} \sin^2 \varphi_1 \sin^3 \varphi_2 d\theta d\varphi_1 d\varphi_2 d\varphi_3 \\ = & \frac{3}{8\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \cos \varphi_3 \sin^4 \varphi_3 \int_0^{\pi/2} E(i \sin \varphi_1 \sin \varphi_2 \tan \varphi_3) \sin^2 \varphi_1 \sin^3 \varphi_2 d\varphi_1 d\varphi_2 d\varphi_3 \\ = & \frac{3}{8\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \sin \varphi_2 \cos \varphi_3 \sin^4 \varphi_3 \frac{f(\varphi_2, \varphi_3) + g(\varphi_2, \varphi_3) + h(\varphi_2, \varphi_3)}{3 \tan^2 \varphi_3} d\varphi_2 d\varphi_3 \end{aligned}$$

where

$$f(\varphi, \psi) = (-2 + 4 \sin^2 \varphi \tan^2 \psi) E \left( i \sqrt{\frac{\sqrt{1 + \sin^2 \varphi \tan^2 \psi} - 1}{2}} \right)^2,$$

$$\begin{aligned} g(\varphi, \psi) &= 2 \left( 1 - 2 \sin^2 \varphi \tan^2 \psi + \sqrt{1 + \sin^2 \varphi \tan^2 \psi} \right) \\ & \quad \times E \left( i \sqrt{\frac{\sqrt{1 + \sin^2 \varphi \tan^2 \psi} - 1}{2}} \right) K \left( i \sqrt{\frac{\sqrt{1 + \sin^2 \varphi \tan^2 \psi} - 1}{2}} \right), \end{aligned}$$

$$h(\varphi, \psi) = \left( -1 + \sin^2 \varphi \tan^2 \psi - \sqrt{1 + \sin^2 \varphi \tan^2 \psi} \right) K \left( i \sqrt{\frac{\sqrt{1 + \sin^2 \varphi \tan^2 \psi} - 1}{2}} \right)^2.$$

Let

$$u = \sqrt{\frac{\sqrt{1+\sin^2 \varphi \tan^2 \psi} - 1}{2}}, \quad v = \sin \varphi$$

then

$$1 + 2u^2 = \sqrt{1 + \sin^2 \varphi \tan^2 \psi}, \quad \sqrt{1 - v^2} = \cos \varphi$$

hence

$$\begin{aligned} \tan^2 \psi &= \frac{(1 + 2u^2)^2 - 1}{v^2} = \frac{4u^2(1 + u^2)}{v^2}, \\ \sec^2 \psi &= 1 + \frac{4u^2(1 + u^2)}{v^2} = \frac{4u^2(1 + u^2) + v^2}{v^2} \end{aligned}$$

and

$$\begin{aligned} 2(1 + 2u^2)(4u) \frac{\partial u}{\partial \psi} &= (\sin^2 \varphi) (2 \tan \psi \sec^2 \psi) \\ &= v^2 \cdot 2 \sqrt{\frac{4u^2(1 + u^2)}{v^2}} \frac{4u^2(1 + u^2) + v^2}{v^2} \\ &= \frac{(4u) [4u^2(1 + u^2) + v^2] \sqrt{1 + u^2}}{v}. \end{aligned}$$

It follows that

$$\frac{\partial u}{\partial \psi} = \frac{[4u^2(1 + u^2) + v^2] \sqrt{1 + u^2}}{2(1 + 2u^2)v}, \quad \frac{\partial v}{\partial \varphi} = \sqrt{1 - v^2}$$

hence

$$\left| \frac{\partial(u, v)}{\partial(\varphi, \psi)} \right| = \frac{[4u^2(1 + u^2) + v^2] \sqrt{1 + u^2} \sqrt{1 - v^2}}{2(1 + 2u^2)v}$$

hence

$$\begin{aligned} \frac{\sin \varphi \cos \psi \sin^4 \psi}{\tan^2 \psi} d\varphi d\psi &= \sin \varphi \cos^3 \psi \sin^2 \psi d\varphi d\psi \\ &= v \left( \frac{4u^2(1 + u^2) + v^2}{v^2} \right)^{-3/2} \left[ 1 - \left( \frac{4u^2(1 + u^2) + v^2}{v^2} \right)^{-1} \right] d\varphi d\psi \\ &= v \frac{v^3}{[4u^2(1 + u^2) + v^2]^{3/2}} \frac{4u^2(1 + u^2)}{4u^2(1 + u^2) + v^2} d\varphi d\psi \\ &= \frac{v^4}{[4u^2(1 + u^2) + v^2]^{5/2}} 4u^2(1 + u^2) \left| \frac{\partial(\varphi, \psi)}{\partial(u, v)} \right| du dv \\ &= \frac{v^4}{[4u^2(1 + u^2) + v^2]^{7/2}} 4u^2 \sqrt{1 + u^2} \frac{2(1 + 2u^2)v}{\sqrt{1 - v^2}} du dv. \end{aligned}$$

From

$$\int_0^1 \frac{v^5}{[4u^2(1+u^2)+v^2]^{7/2}} \frac{1}{\sqrt{1-v^2}} dv = \frac{4}{15} \frac{1}{(1+2u^2)^6} \frac{1}{u\sqrt{1+u^2}}$$

we deduce that the original integral  $\zeta_5/640$  is equal to

$$\begin{aligned} & \frac{4}{15\pi^2} \int_0^\infty \frac{[-2 + 4((1+2u^2)^2 - 1)] E(iu)^2}{(1+2u^2)^5} u du \\ & + \frac{8}{15\pi^2} \int_0^\infty \frac{[1 - 2((1+2u^2)^2 - 1) + (1+2u^2)] E(iu) K(iu)}{(1+2u^2)^5} u du \\ & + \frac{4}{15\pi^2} \int_0^\infty \frac{[-1 + ((1+2u^2)^2 - 1) - (1+2u^2)] K(iu)^2}{(1+2u^2)^5} u du \end{aligned}$$

which is  $(2/5)(\zeta_4/256)$ .

A more complicated but similar argument yields that  $\zeta_6 = \zeta_4$ . Applying the same procedure for  $n = 3$ , however, is fruitless. The hard integral in this special case becomes

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\pi/2} \sin \varphi \sqrt{\sin^2 \theta \sin^2 \varphi + \cos^2 \varphi} \frac{1}{4\pi} \sin \varphi d\varphi d\theta \\ & = \int_0^{\pi/2} \int_0^{\pi/2} \cos \varphi \sin^2 \varphi \sqrt{\sin^2 \theta \tan^2 \varphi + 1} \frac{1}{4\pi} d\theta d\varphi \\ & = \int_0^{\pi/2} \cos \varphi \sin^2 \varphi E(i \tan \varphi) \frac{1}{4\pi} d\varphi \\ & = \frac{1}{4\pi} \int_0^\infty \frac{t^2 E(it)}{(1+t^2)^{5/2}} dt = \frac{1}{96} \zeta_3 \end{aligned}$$

which bears no obvious resemblance to  $(8/3)(\zeta_4/256)$ . Therefore  $\zeta_3 = \zeta_4$  remains unproven, implying that  $\zeta_4 = 3\pi {}_3F_2(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; 1)$  is not yet a theorem.

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